Comp. & Maths. with Appls., Vol. 5, pp. 329–336
Pergamon Press Ltd., 1979. Printed in Great Britain

RATIONAL INTERPOLATION OF WACHSPRESS ERROR ESTIMATES

D. APPRATO, R. ARCANGELI, and J. L. GOUT
Département de Mathématiques, Université de Pau, France

Communicated by E. L. Wachspress

(*Received July* 1979)

1. INTRODUCTION

Wachspress finite elements pose analogous problems to those posed by usual finite elements: for example, interpolation error estimates, the effect of numerical integration in the approximation of elliptic problems and the implementation of these elements. Here we shall discuss only the first of these three points, an approach to the second being described in Apprato–Arcangeli[1] and an example of the implementation of Wachspress finite elements used in Apprato[2].

The problem of interpolation error estimates on polynomial or isoparametric finite elements was resolved by Ciarlet–Raviart[3, 4] and Strang[5]. Methods [3] and [4] which consist of making such estimates, for a straight or curved finite element on a straight reference finite element, do not apply for Wachspress rational finite elements. All our results (see [6–8]) are deduced from general interpolation result Arcangeli–Gout[9] □

It must be seen that a rational type interpolation does not lead to a convergent approximation for usual topologies. Consider the following example: let $[a, b] \subset \mathbf{R}$ and $n \in \mathbf{N}^*$. Let $h = (b-a)/n$ and

$$\forall i = 0, \ldots, n \quad x_i = a + ih.$$

Let p_i, $i = 0, 1, \ldots, n$, be functions defined by: for each $i = 0, 1, \ldots, n$, p_i vanishes out of $[x_i - h, x_i + h] \cap [a, b]$,

$$\forall x \in [a, x_1], \quad p_0(x) = \frac{x_1 - x}{h + x - a},$$

$$\forall x \in [x_{n-1}, b], \quad p_n(x) = \frac{x - x_{n-1}}{h + b - x},$$

and for each $i = 1, \ldots, n-1$

$$\forall x \in [x_{i-1}, x_i], \quad p_i(x) = \frac{x - x_{i-1}}{h + x_i - x},$$

$$\forall x \in [x_i, x_{i+1}], \quad p_i(x) = \frac{x_{i+1} - x}{h + x - x_i}.$$

It is obvious that,

$$\forall i, j = 0, \ldots, n, \quad p_i(x_j) = \delta_{ij}$$

and therefore, each continuous function u on $[a, b]$ has the rational interpolate

$$\Pi u = \sum_{i=0}^{n} u(x_i) p_i.$$

Then, if u defines the function: $x \mapsto 1$ and if we define δ_h by

$$\delta_h = u - \sum_{i=0}^{n} u(x_i)p_i,$$

it is easy to show that

$$\int_a^b \delta_h^2(x)\,dx \geq \frac{b-a}{36},$$

which proves that, when $h \to 0$, δ_h does not tend to 0 in the norm $L^2([a, b])$ or, consequently, in the norm $C^0([a, b])$.

If, for each $i = 0, \ldots, n$, we introduce the functions

$$w_i = \frac{p_i}{\sum_{i=0}^{n} p_i},$$

and if u is the function $x \mapsto x$ and δ_h the function $u - \sum_{i=0}^{n} u(x_i)w_i$, we also prove that $\int_a^b \delta_h'^2(x)\,dx$ does not tend to 0, so that δ_h does not tend to 0 in the norm $H^1([a, b])$.

In fact, we have shown that, for straight Wachspress finite elements, the condition (standard for straight polynomial finite elements)

$$P_K \supset P_k(K),$$

where K is a closed convex polygon, P_K the interpolation space of the finite element constructed over K and $P_k(K)$ the vectorial space formed by the restrictions to K of the kth-degree polynomial functions with 2 variables, leads to the same asymptotic error estimates as those found with usual finite elements of the same degree. It should be noted that, in the previous example, the inclusion $P_K \supset P_0(K)$ (resp. the inclusion $P_K \supset P_1(K)$) where $K = [x_i, x_{i+1}]$ is not proved. □

We shall here consider only straight finite elements in $\mathbf{R}^2$, which are most practical, although the introduction of curved elements (Wachspress[10] for the general construction of rational curved elements and Apprato[2] for results concerning interpolation error) is desirable in theory and sometimes necessary in practice. All considered elements are of class C^0 (in the Ciarlet sense[11]). With one exception they are Lagrange type finite elements.

Using Ciarlet's definition[11], a finite element is a triplet (K, P_K, Σ_K): K always being a closed convex polygon, $\{w_i\}$ denotes the set of basis functions of the finite element; thus P_K is the vectorial space generated by the functions w_i; Σ_K is the set of degrees of freedom for the finite element (for finite elements of Lagrange type, Σ_K is also the set of nodes). We take Wachspress' definition[10], that a finite element is of kth-degree if

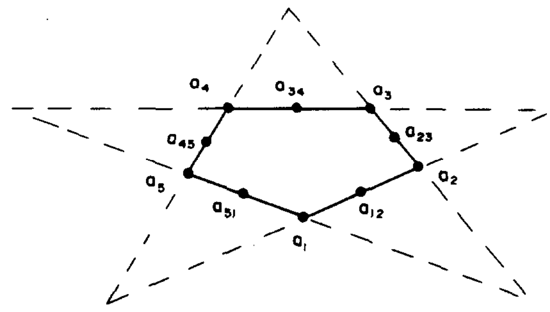
$$P_K \supset P_k(K) \quad \text{and} \quad P_K \not\supset P_{k+1}(K).$$

For each closed convex polygon K with N nodes, $N \geq 4$ having vertices a_i, $i \in I = \mathbf{Z}/N\mathbf{Z}$, numbered such that a_i and a_{i+1} are consecutive, and for each $i \in I$, l_i denotes an element of $P_1(\mathbf{R}^2)$ such that $l_i(x) = 0$ is an equation of the straight line passing through the points a_{i-1} and a_i. Finally h_K (resp. ρ_K) denotes the diameter of K (resp. the maximum diameter of the circles inscribed in K) and $\mathbf{R}^2$ has the euclidean distance δ.

2. WACHSPRESS STRAIGHT FINITE ELEMENTS

The different finite elements introduced in this paragraph correspond to different choices of the convex polygon K: in the cases studied, K is a pentagon, hexagon or a quadrilateral that cannot be transformed into a trapezium or a parallelogram.

2.1 *The pentagon*



Let $q \in P_2(\mathbf{R}^2)$ be such that $q(x) = 0$ is an equation of the conic passing through the five diagonal points (intersection of nonconsecutive sides). When K has one pair (resp. two pairs) of parallel sides, the corresponding diagonal points are at infinity and we always obtain a unique conic which satisfies these conditions. It is clear that, whatever the geometry of K, the conic $q(x) = 0$ does not intersect K.

Then, if P_K is the vectorial space generated by the functions w_i: $K \to \mathbf{R}$, $i \in I$, which are defined by

$$w_i = c_i \frac{l_{i+2} l_{i+3} l_{i+4}}{q} \tag{2.1}$$

where c_i is a constant such that $w_i(a_i) = 1$, and if

$$\sum_K = \{a_i, i \in I\},$$

we show (Wachspress[10]) that the triplet (K, P_K, Σ_K) which is defined by (2.1) and (2.2), is a finite element of first degree: It is the *1st-degree Wachspress pentagonal finite element.* The functions w_i are the basis functions of the finite element.

Likewise, if P_K is now the vectorial space generated by functions w_i and $w_{i-1,i}$: $K \to \mathbf{R}$, $i \in I$, defined by

$$\begin{aligned} w_i &= c_i \frac{l_{i+2} l_{i+3} l_{i+4} l'_i}{q} \\ w_{i-1,i} &= c_{i-1,i} \frac{l_{i+1} l_{i+2} l_{i+3} l_{i+4}}{q} \end{aligned} \tag{2.3}$$

(where $l'_i \in P_1(\mathbf{R}^2)$ is such that $l'_i(x) = 0$ is an equation of the straight line passing through the mid-points $a_{i-1,i}$ and $a_{i,i+1}$ of the sides $a_{i-1}a_i$ and $a_i a_{i+1}$, and where the constants c_i and $c_{i-1,i}$ are such that $w_i(a_i) = w_{i-1,i}(a_{i-1,i}) = 1$), and if

$$\sum_K = \{a_i, a_{i-1,i}; \quad i \in I\}, \tag{2.4}$$

we find again that the triplet (K, P_K, Σ_K) defined by (2.3) and (2.4) is a *2nd-degree* finite element: It is the *2nd-degree Wachspress pentagonal finite element.*

2.2 *The hexagon*

In this case q is an element of $P_3(\mathbf{R}^2)$ such that $q(x) = 0$ is the equation of the cubic which passes through the nine diagonal points. It is possible that the diagonal points are at infinity, but in such cases we also get a unique cubic which satisfies the same conditions. Note that, whatever the geometry of K, the cubic does not intersect K.

First we shall consider that the diagonal points of K stay at finite range: in this case, we find as for the pentagon rational finite elements on the hexagon. The 1st-degree hexagonal finite element of Wachspress is the finite element (K, P_K, Σ_K), where P_K is the vectorial space generated by the functions, from K to $\mathbf{R}$,

$$w_i = c_i \frac{l_{i+2}l_{i+3}l_{i+4}l_{i+5}}{q} \tag{2.5}$$

where the constant c_i is such that $w_i(a_i) = 1$ and where the set Σ_K is defined by

$$\Sigma_K = \{a_i, i \in I\}. \tag{2.6}$$

We define in an analogous manner the 2nd-degree Wachspress hexagonal finite element by

$$\begin{aligned} w_i &= c_i \frac{l_{i+2}l_{i+3}l_{i+4}l_{i+5}l'_i}{q} \\ w_{i-1,i} &= c_{i-1,i} \frac{l_{i+1}l_{i+2}l_{i+3}l_{i+4}l_{i+5}}{q} \end{aligned} \tag{2.7}$$

(where l'_i has the same significance as in (2.1) and where the constants c_i and $c_{i-1,i}$ are such that $w_i(a_i) = w_{i-1,i}(a_{i-1,i}) = 1$), and Σ_K by

$$\Sigma_K = \{a_i, a_{i-1,i};\quad i \in I\}. \tag{2.8}$$

When the diagonal points of K are at infinity, the previous construction is valid unless the cubic $q(x) = 0$ factorizes into a conic and the straight line at infinity. This is the particular case of a *regular hexagon* and has obvious practical value: the relations (2.7) and (2.8) are always valid, but q here is an element of $P_2(\mathbf{R}^2)$ such that $q(x) = 0$ is the equation of the circle passing through the six diagonal points at finite distance. □

The above results can be generalized in the following two ways.

As each closed convex polygon with N sides has $(N(N-3)/2)$ diagonal points and $\dim P_{N-3}(\mathbf{R}^2) = (N(N-3)/2) + 1$, we see that there exists a unique algebraic curve of $(N-3)$th-degree which passes through the $(N(N-3)/2)$ diagonal points and does not intersect K, since each side of K passes through $N-3$ diagonal points of the curve and no other points of the curve.

This result shows that rational finite elements of 1st-and 2nd-degree may be constructed on a ***convex polygon with N sides*** $(N \geq 4)$. But the situation quickly becomes more complicated from a practical point of view as N increases.

On the other hand, it is possible to construct rational finite elements of degree $k > 2$ on a convex polygon with N sides $(N \geq 4)$. To achieve this, it is necessary to have nodes on each side and perhaps interior nodes to determine a polynomial of kth-degree in one variable. We shall not consider, except for the quadrilateral case, the problem of construction of kth-degree finite elements $(k > 2)$.

We refer to Wachspress[10] on these two points.

2.3 *The quadrilateral*

The study of the quadrilateral is very interesting for the following reasons (in particular):

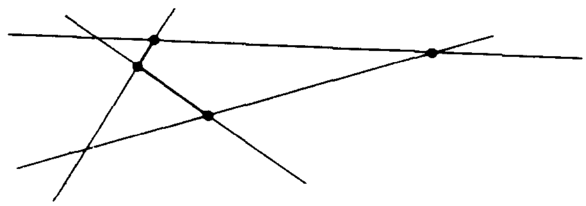
—the line which passes through the diagonal points is the exterior diagonal of the quadrilateral.

—There is a homographic type bijection (therefore a C^∞-diffeomorphism) that changes a square, for example the square $\hat{K} = [-1, +1] \times [-1, +1]$ into any convex quadrilateral K, except trapeziums or parallelograms. The Coxeter–Wachspress mapping F_K [10] is essential for resolving certain difficulties.

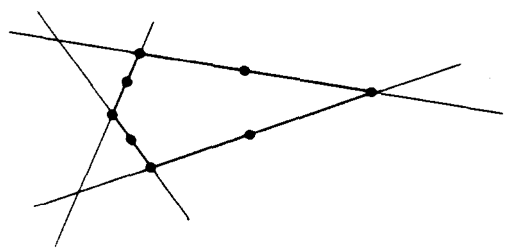
In the following considerations, K is a quadrilateral that is not a trapezium or parallelogram and l is an element of $P_1(\mathbf{R}^2)$ such that $l(x) = 0$ is the equation of the exterior diagonal.

Different quadrilateral finite elements can be constructed. We shall here summarize their definition with a figure indicating the degrees of freedom (the *point* corresponds to the value of the functions, the *circle* to the first derivative of the functions).

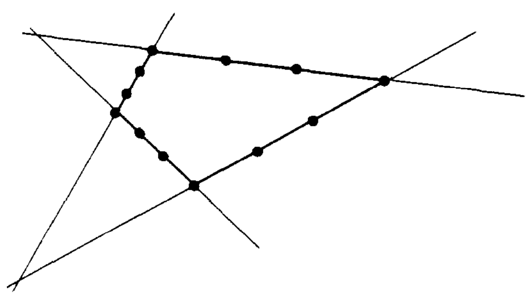
serendipity 1st-degree rational finite element (Wachspress[10])



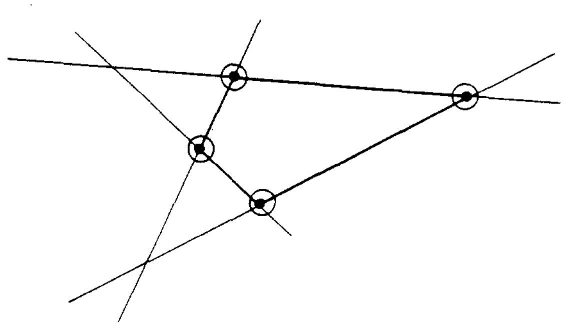
serendipity 2nd-degree rational finite element (Wachspress[10])



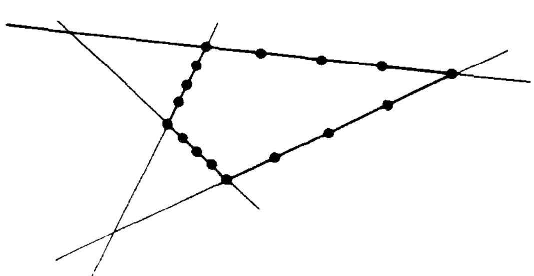
serendipity 3rd-degree rational finite element (Apprato–Arcangeli–Gout[7])



Hermite 3rd-degree rational finite element (Gout[12])



serendipity 4th-degree rational finite element (Gout, in preparation)



The nodes are, according to the cases, the vertices, the points a quarter, a third, a half, two thirds or three quarters along the sides, or the intersection of interior diagonals.

For any of these finite elements, the basis functions w_i are functions: $K \mapsto \mathbf{R}$ of the following expression

$$w_i = c_i \frac{\Psi_i}{l}$$

where c_i is an appropriate constant and $\Psi_i \in P_{k+1}(K)$, k is the degree of the finite element considered.

Only the fourth element is of the Hermite type, the others being of the Lagrange type. The serendipity terminology is justified by the analogy of these elements with the family of serendipity polynomial finite elements over a rectangle (Zienkiewicz[13]). The Hermite element of degree 3 is the rational analogue of Adini's polynomial finite element on a rectangle.

The three last elements were constructed using the Coxeter–Wachspress mapping. Note that these elements have a *minimal number* of nodes (with respect to their degree) Wachspress showed[10] that it is possible to construct finite elements of degree ≥ 3 which have the minimal number of border nodes and a non minimal number of interior nodes. These elements can be of particular interest in certain respects.

3. INTERPOLATION ERROR ESTIMATES

Let K be a polygon. For each $m \in \mathbf{N}$ and for each $p \geq 1$, $W_{m,p}(K)$, instead of $W^{m,p}(\mathring{K})$, is the Sobolev space equipped with semi-norms

$$|v|_{l,p,K} = \left(\sum_{|\alpha|=l} \int |\partial^\alpha v(x)|^p \, \mathrm{d}x\right)^{1/p}, \quad l = 0, 1, \ldots, m$$

when $p < +\infty$, with the usual change when $p = +\infty$.

Let $\mathcal{F}$ be any family of rational finite elements of the same type, denoted (K, P_K, Σ_K) or briefly K. According to the case, the generic element of the family will be:

(i) The serendipity rational finite element of degree 1, 2 or 3.

(ii) The Hermite rational finite element of degree 3.

(iii) The Wachspress pentagonal finite element of degree 1.

(iv) The Wachspress hexagonal finite element of degree 1.

For each $K \in \mathcal{F}$, let q_K be the element of $P_1(\mathbf{R}^2)$ in cases (i) and (ii), $P_2(\mathbf{R}^2)$ in case (iii) and $P_3(\mathbf{R}^2)$ in case (iv). Then q_K becomes l in cases (i) and (ii) and q in cases (iii) and (iv). $\mathcal{F}$ is such that

$$\exists \sigma > 0, \quad \forall K \in \mathcal{F}: \quad \frac{h_K}{\rho_K} \leq \sigma, \tag{3.1}$$

$$\exists \nu > 0, \quad \forall K \in \mathcal{F}: \quad \frac{|q_K|_{0,\infty,K}}{\inf_{x \in K} |q_K(x)|} \leq \nu. \tag{3.2}$$

The condition (3.1) is the standard condition of "no oblateness". The condition (3.2) is an assumption of "non degeneracy". We can note that in cases (i) and (ii), the condition (3.2) is equivalent to

$$\exists \nu_1 > 0, \quad \forall K \in \mathcal{F}: \quad \inf_{x \in K} \delta(x, d) \geq \nu_1 h_k.$$

The fundamental result which concerns interpolation error estimates is, for Lagrange interpolation, a theorem of Arcangeli–Gout (see [9], which is of the following form.

THEOREM 3.1

Let (K, P_K, Σ_K) be a rational finite element of type (i), (iii) or (iv), of degree $k \leq 3$. Let Π_K be the P_K-interpolation operator with respect to Σ_K and $\{w_i\}_{i=1,\ldots,N_0}$ the set of basis functions. Assuming that $k + 1 > (2/p)$, then there exists a constant C independent of h and K such that, for each $v \in W^{k+1,p}(K)$ and for each $m = 0, 1, \ldots, k$ we obtain

$$|v - \Pi_K v|_{m,p,K} \leq C\left(\sum_{i=1}^{N_0} |w_i|_{m,\infty,K}\right)|v|_{k+1,p,K}{h_K}^{k+1-m}.$$

There is[14] an analogous theorem for Hermite interpolation, necessary when dealing with case of the type (ii) element. From this, the general result of interpolation error which concerns rational finite elements can be deduced.

THEOREM 3.2

Let $(K, P_K, \Sigma_K) \in \mathcal{F}$ be a rational finite element of degree $k \leq 3$ and let Π_K be the P_K-interpolation operator with respect to Σ_K. If $k+1 > (2/p)$ and $\mathcal{F}$ satisfies (3.1) and (3.2), then, for each $m = 0, 1, \ldots, k$ there exists a constant C independent of K such that, for each $v \in W^{k+1,p}(K)$, we obtain

$$|v - \Pi_K v|_{m,p,K} \leq C|v|_{k+1,p,K}{h_K}^{k+1-m}.$$

Proof

We refer to Apprato–Arcangeli–Gout[7] for elements of type (i), Gout ([8] and [to appear]) for elements of type (ii), (iii) and (iv).

The key idea of the proof is as follow: first we obtain results for $m = 0$; when $k = 1$ the result is obvious since

$$\sum_{i=1}^{N_0} |w_i|_{0,\infty,K} \leq N_0,$$

but geometrical considerations are necessary when $k = 2$ or 3. The Coxeter–Wachspress mapping plays a major part in this proof.

Using Wilhelmsen's result[15] we show that there exists a constant C such that

$$\forall m = 1, \ldots, k, \quad \forall i = 1, \ldots, N_0, \quad |w_i|_{m,\infty,K} \leq \frac{C}{{\rho_K}^m}|w_i|_{0,\infty,K}.$$

Using this "inverse inequality" the proof can be completed.

Remarks

1. Wachspress rational interpolation therefore constitutes in all the cases studied, a *convergent approximation in the norm* $W^{1,p}$.

2. Rational finite elements of types (i), (ii), (iii) and (iv) give the same asymptotic errro estimates as those obtained for usual finite elements of the same degree.

3. It can be shown that these error estimates are optimal.

4. The interpolation error estimates of the 2nd-degree pentagonal (resp. hexagonal Wachspress finite element and of the 4th-degree serendipity rational finite element are being studied. We think that the semi-norm $| \quad |_{m,p,K}$ of the corresponding interpolation error will also be in $O({h_K}^{k+1-m})$.

REFERENCES

1. D. Apprato and R. Arcangeli, Approximation d'un problème aux limites elliptiques d'ordre deux par élément finis rationnels de Wachspress avec intégration numérique. *R.A.I.R.O.* **19**(1), 3–20 (1979).
2. D. Apprato, Thèse 3ème cycle, Pau (1978).
3. P. G. Ciarlet and P. A. Raviart, General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications to finite element methods. *Arch. Rat. Mech. Anal.* **46**, 177–199 (1972).
4. P. G. Ciarlet and P. A. Raviart, Interpolation theory over curved elements with applications to finite element methods. *Comp. Math. Appl. Mech. Engng* **1**, 217–249 (1972).
5. G. Strang, Approximation in the finite element method. *Numer. Math.* **19**, 81–98 (1972).
6. R. Arcangeli, J. L. Gout and R. Royer, Etude de l'erreur d'interpolation rationnelle de Wachspress sur un polygone. Publications Mathématiques de Pau (1976).
7. D. Apprato, R. Arcangeli and J. L. Gout, Sur les éléments finis rationnels de Wachspress. *Numer. Math.* **32**, 247–270 (1979).

8. J. L. Gout, Interpolation error estimates on Hermite rational "Wachspress type" 3rd-degree finite element. *Comput. Maths Applics* **5**, 249–257 (1979).
9. R. Arcangeli and J. L. Gout, Sur l'évaluation de l'erreur d'interpolation de Lagrange dans un ouvert de $\mathbf{R}^n$. *R.A.I.R.O.* **10**(3), 5–27 (1976).
10. E. L. Wachspress, *A Rationel Finite Element Basis.* Academic Press, New York (1975).
11. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems.* North Holland, Amsterdam (1977).
12. J. L. Gout, Construction of a Hermite rational "Wachspress type" finite element. *Comput. Maths. Applics.* **5**, 337–347 (1979).
13. O. C. Zienkiewicz, *La Méthode des Éléments Finis.* Ediscience, Paris (1973).
14. J. L. Gout, Estimation de l'erreur d'interpolation d'Hermite dans $\mathbf{R}^n$. *Numer. Math.* **28**, 407–429 (1977).
15. R. Wilhelmsen, A Markov inequality in several dimensions. *J. Approx. Theory* **11**, 216–220 (1974).
16. E. L. Wachspress, A rational basis for function approximation. *Proc. Conf. of Appl. Numerical Anal. Dundee*, Lecture Notes in Math, Vol. 228, pp. 223–252. Springer Verlag, Berlin (1971).
17. E. L. Wachspress, A rational basis for function approximation. Part II, Curved sides. *J. Inst. Math. Appl.* **11**, 83–104 (1973).
18. E. L. Wachspress, Algebraic geometry foundations for finite element computation. *Conf. Numerical Sol. Diff. Eqs. Dundee*, Springer-Verlag Lecture Notes in Mathematics, Vol. 363, pp. 177–188. Springer-Verlag, Berlin (1974).